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Quaternionic-like bifurcation in the absence of symmetry

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Abstract. We give some results concerning the existence of bifurcating periodic solutions of non-linear time evolution equations with the property that criticality is produced by a couple of imaginary eigenvalues with double multiplicity. This situation is also compared with the classical Hopf case and with the 'quaternionic bifurcation' occurring in the presence of a SU(2) symmetry.

1. Introduction

It is well known that the Hopf bifurcation is the simplest mechanism which has the appearance of periodic motion bifurcating from stationary solutions of non-linear evolution equations. The typical condition for the occurrence of this phenomenon is that, for some critical value λ_0 of one 'control parameter' λ, two eigenvalues of the linearised part of the equations cross the imaginary axis (see [1 − 5]).

In this paper we present some results generalising this problem: we will deal with the existence of bifurcating periodic solutions of equations of the following type

$$\dot{x} = f(\lambda, x) \tag{1}$$

where

$$x \in R^n, x = x(t) \qquad \lambda \in R^p$$

with the main property that criticality is given by two imaginary eigenvalues with *double* multiplicity.

A special case for this situation occurs when one can reduce the original problem (1) into a four-dimensional equation which exhibits a covariance property under a particular representation T of the symmetry group SU_2. This case has been already considered [6, 7] and will be briefly mentioned here in § 3: due to its peculiar group-theoretical structure, it gives rise to a specific bifurcation which is called the 'quaternionic bifurcation'. For this reason, and because the group-theoretical frame reveals it to be useful even when the SU_2 covariance is not verified, we have called the 'quaternionic-like' bifurcation this general case 'without symmetry', which will be considered in this paper.

2. Statement of the problem

Let us consider (1), where

$$f: R^p \times R^n \to R^n \qquad \text{and} \qquad f(\lambda, 0) = 0$$

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with usual regularity assumptions, and suppose that for some value $\lambda = \lambda_0 \equiv (\lambda_{01}, \ldots, \lambda_{0p})$ of the control parameters the linearised part of f (the prime means differentiation)

$$L(\lambda) \equiv f'_x(\lambda, 0) \tag{2}$$

possesses two complex conjugated imaginary eigenvalues, say $\pm i\omega_0(\omega_0 > 0)$, each with multiplicity two (and then the corresponding real eigenspace is four dimensional).

Assumption A. The eigenvalues $\pm i\omega_0$ of $L(\lambda_0)$ have both algebraic and geometrical multiplicity equal to 2. No other eigenvalue of $L(\lambda_0)$ is a multiple of $\pm i\omega_0$.

After introducing, as usual, a new real parameter ω in such a way that, rescaling the time variable t:

$$t \to \tau = \omega t \tag{3}$$

(with $\omega = \omega_0$ when $\lambda = \lambda_0$) one has to look for solutions with period 2π, let us embed the functions $x(\tau)$ in the space $L^2((0, 2\pi), R^n)$ equipped with the scalar product

$$(x(\tau), y(\tau))_{L^2} = \int_0^{2\pi} d\tau (x(\tau), y(\tau))_{R^n}.$$

Assumption A implies then first that, by means of a linear transformation in R^n, the operator $L_0 = L(\lambda_0)$ can be put in the form

$$L_0 = L(\lambda_0) = \left(\begin{array}{c|c} \omega_0 K_1 & 0 \\ \hline 0 & \mathscr{L}_0 \end{array}\right)\}4 \qquad K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{4}$$

and that the kernel V in $L^2((0, 2\pi), R^n)$ of the operator

$$M_0 \equiv M(\lambda_0) = \omega_0\, d/d\tau - L(\lambda_0)$$

is the four-dimensional real space spanned by the vectors

$$e_1 = \begin{bmatrix} \cos\tau \\ -\sin\tau \\ 0 \\ 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} \sin\tau \\ \cos\tau \\ 0 \\ 0 \end{bmatrix} \qquad e_3 = \begin{bmatrix} 0 \\ 0 \\ \cos\tau \\ -\sin\tau \end{bmatrix} \qquad e_4 = \begin{bmatrix} 0 \\ 0 \\ \sin\tau \\ \cos\tau \end{bmatrix} \tag{5}$$

(clearly, this is the explicit form of the vectors in the basis in which $L(\lambda_0)$ has the form (4), and the remaining $(n-4)$ components equal to zero are neglected). It can be useful also to remark that one can also write, e.g.,

$$\begin{aligned} e_1 &= \mathrm{Re} \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} e^{i\tau} = \exp(K_1\tau) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ e_2 &= \mathrm{Im} \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} e^{i\tau} = \exp(K_1\tau) \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = -\exp[K_1(\tau + \pi/2)] \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = -K_1 e_1 \end{aligned} \tag{6}$$

in particular $\exp(K_1\tau) = I\cos\tau + K_1\sin\tau$. Again, assumption A shows that V is also the kernel of the formal adjoint

$$M_0^+ = -\omega_0 \mathrm{d}/\mathrm{d}\tau - L_0^+$$

and that

$$L^2((0, 2\pi), R^n) = V \oplus W$$

where W is the range of M_0. Then standard Lyapunov-Schmidt procedure [1-3] can be used, and writing $x(\tau) = v(\tau) + w(\tau)$, with $v \in V$, $w \in W$, one obtains, from the projection on W of (1), that w is a function of λ and v

$$w = w(\lambda, v)$$

such that, as is well known,

$$w'_\lambda(\lambda_0, 0) = w'_v(\lambda_0, 0) = 0.$$

This allows us to neglect—at least when we analyse only terms which are linear in $(\lambda - \lambda_0)$ and v—the $(n-4)$ components $x_5(\tau), \ldots, x_n(\tau)$ in (1), and then reduce the problem (1) to the subspace $L^2((0, 2\pi), R^4)$, namely to consider the following problem:

$$\omega\, \mathrm{d}u/\mathrm{d}\tau = f(\lambda, u) \qquad f: R^p \times R^4 \to R^4 \qquad u \in R^4 \tag{6'}$$

(having also used (3)), with in particular

$$f'_u(\lambda_0, 0) = L(\lambda_0) = \omega_0 K_1 \tag{6''}$$

where we have used the same notation for f and $L(\lambda)$ as above also for the quantities reduced to this subspace.

We shall examine in the following some cases in which the existence of bifurcating periodic solutions of (6) is ensured.

3. Bifurcation theorems: I. The SU_2 symmetric case

An hypothesis which ensures the existence of bifurcated periodic solutions has already been considered [6-7]. For completeness, and in view of the following discussion, let us briefly recall some relevant properties of this case. The hypothesis amounts essentially to assuming that the original problem can be suitably reduced to the following form

$$\dot{u} = F(\lambda, u) \qquad u = u(t) \qquad u \in R^4,\ \lambda \in R \tag{7}$$

in such a way that the map $F: R \times R^4 \to R^4$ is covariant

$$F(\lambda, T(g)u) = T(g)F(\lambda, v)$$

with respect to the four-dimensional real irreducible representation T of the group SU_2, which is generated by the operators $\frac{1}{2}H_i$ $(i = 1, 2, 3)$:

$$H_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad H_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{8}$$

As discussed in references [6, 7], this is a 'quaternionic type representation' [8]: in fact, the centre of T, i.e. the space of the operators commuting with T (and then with H_i) is four dimensional. As a basis for this space, we choose the following 'intertwining operators'

$$K_0 = I \qquad K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
$$K_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{9}$$

As a consequence of covariance, one has in particular that (the sum is implicit over repeated indices $\alpha = 0, 1, 2, 3$)

$$L(\lambda) = \mu_\alpha(\lambda) K_\alpha \tag{10}$$

which in fact has double eigenvalues, given by

$$\sigma(\lambda) = \mu_0(\lambda) \pm \mathrm{i}\left(\sum_i \mu_i^2(\lambda)\right)^{1/2}$$

and the case considered in this paper is obtained if $\mu_0(\lambda_0) = 0$ and $\mu_i(\lambda_0) \neq 0$ for at least one index $i = 1, 2, 3$. Referring to [6, 7] for details, one sees that in this case the problem has bifurcating periodic solutions, which can be written in the form (here and in the following, sum over repeated indices $i = 1, 2, 3$)

$$u(t) = r \exp(\nu_i K_i t)\hat{v} = r\left(I \cos \omega t + \frac{\sin \omega t}{\omega} \nu_i K_i\right)\hat{v} \tag{11}$$

where $\hat{v}$ is any unit vector in R^4, ν_i are suitable functions of

$$r^2 = (u(t), u(t))_{R^4} = \text{constant} \tag{12}$$

and

$$\omega = (\nu_i \nu_i)^{1/2}. \tag{13}$$

In this problem, of course, only one control parameter λ (i.e. $p = 1$) is needed, but three other real parameters ν_i are naturally introduced to describe the solution.

4. Some remarks on covariance properties and the standard Hopf problem

The situation considered in the previous section can be compared with the classical Hopf bifurcation problem: it is known in fact [9, 10] that the intrinsic SO_2 covariance of the Hopf problem produced by time translations

$$\tau \to \tau + \tau' \qquad (\text{mod } 2\pi)$$

and the periodicity requirement consequently means that the equation obtained by means of a standard Lyapunov–Schmidt procedure displays a 'temporal' [10] SO_2 covariance, operating in the real two-dimensional kernel of the linearised problem.

In the general four-dimensional case, instead, we are not able to give *a priori* some dynamical condition which could induce a SU_2 covariance: in fact, in the above section, this covariance was introduced as an explicit assumption. This suggests the problem of examining the possible existence of bifurcating solutions even if this covariance requirement is eliminated (or suitably substituted).

Actually, there are some other strong differences between the Hopf case and the present one, which will emerge in the following, and which make the problem more difficult.

First of all, let us write the Hopf equation in a form analogous to (6), having used similar arguments (and the same notation):

$$\omega \, du/d\tau = f(\lambda, u) \qquad u \in R^2 \qquad f: R \times R^2 \to R^2$$

where now

$$L(\lambda_0) = f'_u(\lambda_0, 0) = \omega_0 J = \omega_0 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Using the Lyapunov-Schmidt method, the linear part of the reduced bifurcation equation has the form, with obvious notation,

$$L_V(\lambda) = a(\lambda) I + b(\lambda) J \qquad a(\lambda_0) = 0 \qquad b(\lambda_0) = \omega_0 \tag{14}$$

as a consequence of the SO_2 covariance already mentioned, and in particular of the property that in this case the centre is two dimensional and generated by I and J. In addition to this fact, let us remark that

(i) the operator J maps the two-dimensional kernel V of $d/d\tau - J$ into itself;
(ii) for any non-zero vector $v \in V$, the vectors Iv and Jv are a basis for V;
(iii) the linear span of the orbit, $\exp(J\tau)\hat{v}$, under time action, of any non-zero vector $\hat{v} \in R^2$ gives the whole space V, so all vectors in this space are equivalent.

5. Preliminary results

Let us return now to (6) and in particular to the linear operator $L(\lambda)$

$$L(\lambda) = f'_u(\lambda, 0)$$

with $L(\lambda_0) = \omega_0 K_1$, and introduce the subspace W_1, orthogonal to the kernel V of $(d/d\tau - K_1)$, of vectors in W having 'frequency 1', i.e. the four-dimensional subspace of W generated by (cf (5))

$$\begin{gathered} f_1(\tau) = \begin{bmatrix} \cos\tau \\ \sin\tau \\ 0 \\ 0 \end{bmatrix} = \exp(-K_1\tau) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = e_1(-\tau) \\ f_2(\tau) = \begin{bmatrix} -\sin\tau \\ \cos\tau \\ 0 \\ 0 \end{bmatrix} \qquad f_3(\tau) = \begin{bmatrix} 0 \\ 0 \\ \cos\tau \\ \sin\tau \end{bmatrix} \qquad f_4(\tau) = \begin{bmatrix} 0 \\ 0 \\ -\sin\tau \\ \cos\tau \end{bmatrix}. \end{gathered} \tag{15}$$

One can show, by means of some simple algebra, the following lemmas.

Lemma 1. The operator $L(\lambda)$ can be written as a sum of two terms

$$L(\lambda) = L_V(\lambda) + \tilde{L}(\lambda)$$

where

$$L_V(\lambda): V \to V \qquad \tilde{L}(\lambda): V \to W_1.$$

The operator $L_V(\lambda)$ can be written as a combination of

$$I,\ K_1,\ H_i,\ K_1 H_i = H_i K_1 \qquad (i = 1, 2, 3)$$

where K_i and H_i are defined in (8) and (9), whereas $\tilde{L}(\lambda)$ is a combination of

$$K_2,\ K_3,\ K_2 H_i,\ K_3 H_i.$$

If the terms proportional to $K_i H_j$ are absent, the eigenvalues of $L(\lambda)$ are purely imaginary, but in general the double multiplicity of the eigenvalues $\pm i\omega_0$ of $L(\lambda_0)$ is removed for $\lambda \neq \lambda_0$. Precisely, the eigenvalues of the combination

$$\mu_0 I + \mu_i K_i + \gamma_i H_i$$

are

$$\mu_0 \pm i(\mu_i \mu_i)^{1/2} \pm i(\gamma_i \gamma_i)^{1/2}.$$

The terms proportional to $K_i H_j$ give a real part to the eigenvalues, which take the form $a \pm i\omega$ and $-a \pm i\omega$.

Lemma 2. For any non-zero vector $\hat{v} \in R^4$, the four vectors $K_\alpha \hat{v}$ $(\alpha = 0, \ldots, 3)$ are linearly independent. The same is true for $\hat{v}$, $H_i \hat{v}$. Instead, there are vectors $v_0 \in R^4$ such that, e.g., $K_1 v_0 = H_1 v_0$.

The last remark in lemma 1 shows that the presence in $L(\lambda)$ of terms $K_i H_j$ can exclude, or at least make problematic, the stability of a possible solution of (6). We then assume, both for simplicity and for providing a necessary condition for stability:

Assumption B. The part $L_V(\lambda)$ of $L(\lambda)$ is a combination of the following type

$$L_V(\lambda) = \alpha(\lambda) I + \omega_0(1 + \eta_1(\lambda)) K_1 + \gamma_i(\lambda) H_i$$

where α, η_1, γ_i are smooth functions of λ vanishing for $\lambda = \lambda_0$.

Let us get also the following useful tool.

Lemma 3. Let $\Phi(\lambda, v)$ be a smooth function

$$\Phi: R^m \times R^m \to R^m$$

with $\Phi(\lambda, 0) = 0$ and, for some $\lambda = \lambda_0$, $\Phi'_v(\lambda_0, 0) = 0$. If there is a (unit) vector $\hat{v} \in R^m$ such that the m vectors defined by

$$\frac{\partial}{\partial \lambda_i} \Phi'_v(\lambda_0, 0)\hat{v} \qquad (i = 1, \ldots, m)$$

are linearly independent, then the equation

$$\Phi(\lambda, v) = 0$$

has a non-zero solution, which can be written

$$v = s\hat{v}$$

$$\lambda_i = \lambda_i(s) \qquad \text{with } \lambda_i(s) \to \lambda_{0i} \qquad \text{for } s \to 0$$

where s is a real parameter defined in a neighbourhood of zero.

Proof. First, we write $\Phi(\lambda, v) = \Psi(\lambda, v)v$, where Ψ is a $m \times m$ matrix, with $\Psi(\lambda_0, 0) = 0$; putting then $v = s\hat{v}$, equation $\Phi = 0$ becomes ($s = 0$ is the trivial solution)

$$\Psi(\lambda, s\hat{v})\hat{v} = 0$$

where $\hat{v}$ is fixed. From the given hypothesis, $(\partial\Psi(\lambda_0, 0)/\partial\lambda_i)\hat{v}$ are independent vectors. Then an easy application of the implicit function theorem gives the result.

6. Bifurcation theorems: II

After this long preparation, let us finally consider (6) and apply to it the standard Lyapunov-Schmidt procedure. As already observed, the kernel of the linearised part $d/d\tau - K_1$ is four dimensional, and then one will find in general four independent equations to be solved: this makes it reasonable that, in addition to the parameter ω, one has to deal with three parameters $\lambda \equiv (\lambda_1, \lambda_2, \lambda_3)$. On the other hand, apart from the SU_2 symmetric case, the general case, in which only one control parameter $\lambda \in R$ is present, has already been considered and solved in detail [11].

It can be useful to compare the present situation with the Hopf case. In particular, note that the SO_2 covariance under time translations gives in the present case only the condition that $L_V(\lambda)$ has to commute with K_1, the operator which plays here the same role as J in the Hopf problem: and just this condition in the Hopf problem was essentially sufficient for determining both the form of the operator L_V (see (14)) and the number of parameters (λ and ω) necessary for solving the reduced equation.

We can now state our main result.

Theorem 4. Let us consider (1), or assume directly (6), together with A; suppose now that $p = 3$, i.e. $\lambda \equiv (\lambda_1, \lambda_2, \lambda_3) \in R^3$. With the notations of lemma 1, let $L_V(\lambda)$ satisfy B. Then, for each unit vector $\hat{v} \in R^4$ such that the following four vectors

$$K_1\hat{v}$$

and

$$\left(\frac{\partial\alpha}{\partial\lambda_j} I + \sum_{i=1}^{3} \frac{\partial\gamma_i}{\partial\lambda_j} H_i\right)\hat{v} \qquad (j = 1, 2, 3)$$

(where the derivatives are evaluated at $\lambda = \lambda_0$) are linearly independent, a periodic solution bifurcates at $\lambda = \lambda_0$, of the following form

$$x(\tau, s) = s\exp(K_1\tau)\hat{v} + w(\tau, s)$$

$$\tau = \omega t$$

$$\omega = \omega_0 + \omega^{(1)}(s)$$

$$\lambda_j = \lambda_{0j} + \lambda_j^{(1)}(s)$$

where s is a real parameter defined in a neighbourhood of zero, and

$$\lim_{s\to 0} s^{-1}w(\tau, s) = \lim_{s\to 0} \omega^{(1)}(s) = \lim_{s\to 0} \lambda_j^{(1)}(s) = 0.$$

Proof. Applying the Lyapunov-Schmidt method to (1), or to (6), and observing that in the kernel V one has $d/d\tau = K_1$, the reduced equation obtained from the projection on V has the form

$$\Phi(\lambda, \omega; v) \equiv \left(-\omega K_1 + L_V(\lambda)\right) v + h(\lambda, \omega; v) = 0$$

having used also lemma 1, and where the higher-order term $h(\lambda, \omega; v)$ is such that

$$h(\lambda, \omega; 0) = 0$$

$$\frac{\partial}{\partial \lambda} h'_v(\lambda_0, \omega_0; 0) = \frac{\partial}{\partial \omega} h'_v(\lambda_0, \omega_0; 0) = 0.$$

Then, if L_V satisfies B, all hypotheses of lemma 3 (with $m = 4$) are verified whenever $\hat{v}$ is chosen as prescribed in the theorem, and this concludes the proof.

Note, finally, that in the three-dimensional manifold of unit vectors $\hat{v}$ in R^4, the orbit $\exp(K_1\tau)\hat{v}$ describes a one-dimensional submanifold; therefore, we can conclude that there is in general a double infinity of independent bifurcating periodic solutions of our equation, which can be obtained choosing different vectors $\hat{v}$ in theorem 4 (see also lemma 2).

As a simple example, in which all the above results can be easily verified, one can consider the case

$$\alpha(\lambda) = \lambda_1 \qquad \gamma_1(\lambda) = 0$$

$$\gamma_2(\lambda) = \lambda_2 \qquad \gamma_3(\lambda) = \lambda_3$$

where all hypotheses of theorem 4 are satisfied, for any $\hat{v} \in R^4$.

The peculiar differences between the results presented above and the classical Hopf case now appear clear: compare especially lemma 1 and theorem 4 with the remarks in § 4.

7. The Lyapunov–Schmidt method in the SU_2 symmetric case

We have seen in lemma 1 that the operators K_2 and K_3 map V into W_1 (and, similarly, W_1 into V), e.g. one has, directly from the definitions, $K_3 e_1 = f_4$ and $K_3 f_4 = -e_1$. Therefore, the linear part of the bifurcation equation obtained via Lyapunov-Schmidt projection cannot contain these operators. What happens then if the original equation is covariant with respect to the group SU_2 as assumed in § 3? In this case, the linear part of this equation has the form given in (10), and so the linear part of the corresponding bifurcation equation obtained by Lyapunov-Schmidt projection is expected to be a combination of I and K_1 only. Actually, one discovers that, in this case, the projected bifurcation equation is *identically satisfied*, if only the condition (see (12))

$$(u, u)_{R^4} = r^2 = \text{constant}$$

is imposed. In fact, let us write the SU_2-covariant equation (7) in the form (we assume without loss of generality that $L(\lambda_0) = \omega_0 K_1$; here and in the following, sum over repeated indices $\alpha = 0, 1, 2, 3$)

$$du/dt = \omega du/d\tau = \omega_0 K_1 u + \eta_\alpha(\lambda) K_\alpha u + h_\alpha(\lambda, r) K_\alpha u \qquad (16)$$

where $\lambda \in R$, η_α vanish for $\lambda \to \lambda_0$, and h_α are given higher-order functions with $h_\alpha(\lambda, 0) = 0$ $(\alpha = 0, \ldots, 3)$. Put

$$\eta_0(\lambda) + h_0(\lambda, r) = 0$$

which is precisely the condition $(u, u) = r^2 = \text{constant}$, and

$$\nu_1 = \omega_0 + \eta_1 + h_1 \qquad \nu_{2,3} = \eta_{2,3} + h_{2,3}.$$

Projecting now (16) on W_1 and V (in this case in fact the whole solution (11) belongs to $V \oplus W_1$, then $u = v + w_1$), and observing that

$$d/d\tau = K_1 \text{ in } V \qquad \text{and} \qquad d/d\tau = -K_1 \text{ in } W_1$$

one has from the projection on W_1 and V respectively

$$-(\omega + \nu_1)K_1 w_1 = (\nu_2 K_2 + \nu_3 K_3)v$$

and

$$(\omega - \nu_1)K_1 v = (\nu_2 K_2 + \nu_3 K_3)w_1.$$

Substituting in the last equation the quantity w_1 taken from the preceding one, the bifurcation equation is

$$(\omega - \nu_1)K_1 v - (\omega + \nu_1)^{-1}(\nu_2^2 + \nu_3^2)K_1 v = 0$$

which is identically satisfied for all v, if $\nu_i \nu_i = \omega^2$, just as in § 3. This is quite different from the Hopf case, due to the fact that there the centre is two dimensional and to the properties listed at the end of § 4.

8. Bifurcation theorems: III

Another question however may be posed. Let SU_2 act on the space V through its representation T already considered: then, there are certainly four intertwining operators K_α^V, mapping V in itself and commuting with this representation: what is the form, written in the space V, of these operators? One can directly obtain this form writing both the group action and the intertwining operators with respect to the basis $\{e_s\}$ $(s = 1, \ldots, 4)$ of V given in (5). Alternatively, one can observe that the operator R of time reversal

$$(Ru)(t) = u(-t) \tag{17}$$

maps V into W_1 and *vice versa*, and obviously commutes with the SU_2 action. Therefore, four operators mapping V in itself and commuting with SU_2 are

$$I, \quad K_1, \quad K_2 R, \quad K_3 R.$$

As a consequence of this fact, one gets the following result.

Theorem 5. Let in (6) $\lambda \in R^4$ and $f'_u(\lambda, 0)u = L(\lambda)u$ be of the following form $(\alpha = 0, \ldots, 3)$

$$L(\lambda)u = K_1 u(t) + \eta_\alpha(\lambda)K_\alpha u(t) + \chi_\alpha(\lambda)K_\alpha u(-t)$$

where η_α, χ_α are given functions of λ vanishing for $\lambda \to \lambda_0$, and suppose that

$$\det\left(\frac{\partial(\eta_0, \eta_1, \chi_2, \chi_3)}{\partial(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}\right) \neq 0.$$

Then, for any unit vector $\hat{v} \in R^4$, there is a bifurcating periodic solution with critical frequency $\omega = 1$ of (6). Alternatively, as in theorem 4, one can assume that there are three parameters λ, and insert as an additional parameter the frequency ω of the solution.

Proof (a sketch). It suffices to observe that, operating by means of the Lyapunov-Schmidt method, the leading terms of the reduced equation are $(\eta_0 I + \eta_1 K_1 + \chi_2 K_2 + \chi_3 K_3)\hat{v}$, and then apply lemmas 2 and 3.

Note of course that the presence of time inversion R actually destroys the covariance of the problem with respect to time translations, but that does not prevent from the existence of periodic solutions, and in fact generates them.

9. On the reduction to normal forms

As a final remark, let us point out another difference between the Hopf problem and the four-dimensional one considered in this paper. This deals with the problem of reducing the original equation into a 'normal form' (see [2]), namely of eliminating, by means of a change of variables, all terms of a given order k. As is well known [2], this elimination is possible provided that none of the quantities

$$\sigma_i - \sum_{j=1}^{n} a_j \sigma_j \tag{18}$$

is zero, where σ_i are the eigenvalues of the linear part, and a_j are n integer non-negative numbers such that $\Sigma a_j = k =$ the order of terms to be eliminated.

In the Hopf case $(n = 2, \sigma_i = \pm i)$, it is known [2] that, starting from a generic problem, one can—up to an arbitrarily large order—eliminate even-order terms and reduce odd-order terms to the form $(u, u)^m (a_m I + b_m J) u$; in this way one obtains as a by-product an equation which exhibits in the new variables a 'spatial' SO_2 covariance (which is not to be confused with the intrinsic 'temporal' SO_2 covariance: see [10] and § 4).

Let us apply the same procedure to (6), i.e. to a generic four-dimensional problem with double imaginary eigenvalues at the critical point: now, in (18) one has $n = 4$, $\sigma_i = \pm i$. It is easily seen that all quadratic terms $u_i u_j$ can be eliminated. Instead, a rapid calculation can show that not all cubic terms can be removed; precisely, using the rules given in [2], one sees that there are 12 independent terms, among the 20 possible cubic terms, which cannot be removed. Among these, there are the four terms $(u, u)\ K_\alpha u$, which transform as 'vectors' under the representation T of SU_2 already defined, but also other terms which do not have this property. In conclusion, even at the third order, the reduction to normal form *cannot* transform the problem into a SU_2-covariant problem.

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